WEAK APPROXIMATION ON DEL PEZZO SURFACES OF DEGREE 1

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Abstract. We study del Pezzo surfaces of degree 1 of the form
$$w^2 = z^3 + Ax^6 + By^6$$
in the weighted projective space $\mathbb{P}_k(1,1,2,3)$, where k is a perfect field of characteristic not 2 or 3 and $A, B \in k^*$. Over a number field, we exhibit an infinite family of (minimal) counterexamples to weak approximation amongst these surfaces, via a Brauer-Manin obstruction.

1. Introduction

Let X be a geometrically integral variety over a number field k. Write Ω_k for the set of places of k, and let k_v be the completion of k at $v \in \Omega_k$. Assume that X has k_v-points at every place v. We say that X satisfies weak approximation if the diagonal embedding
$$X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v)$$
is dense for the product of the v-adic topologies. If X' is another k-variety, k-birational to X, and both X and X' are smooth, then X' satisfies weak approximation if and only if X does. As Swinnerton-Dyer puts it, the "dramatic" failure of weak approximation, that is, when $X(k) = \emptyset$ and yet $X(k_v) \neq \emptyset$ for every place v, is referred to as the failure of the Hasse principle; see [Har04].

A del Pezzo surface X is a smooth projective geometrically rational surface with ample anticanonical class $-K_X$. The degree d of X is K_X^2; it is an integer in the range $1 \leq d \leq 9$. When $d \geq 5$, X is known to satisfy both the Hasse principle and weak approximation. On the other hand, there are counterexamples to both of these phenomena when d is 2, 3 and 4, all of which can be explained by a Brauer-Manin obstruction; see [KT04], [CG66] and [BSD75], respectively. Del Pezzo surfaces of degree 1 satisfy the Hasse principle because they come furnished with a rational point: the base-point of the anticanonical linear system. (We refer to this point as the anticanonical point.) We will give examples of these surfaces that do not satisfy weak approximation, following ideas of Kresch and Tschinkel [KT04].

Let k be a perfect field and let $k[x, y, z, w]$ be the weighted graded ring where the variables x, y, z, w have weights $1, 1, 2, 3$, respectively. Set $\mathbb{P}_k(1,1,2,3) := \operatorname{Proj} k[x, y, z, w]$. Every del Pezzo surface of degree 1 over k is isomorphic to a smooth sextic hypersurface in $\mathbb{P}_k(1,1,2,3)$.

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Conversely, any smooth sextic in $\mathbb{P}_k(1,1,2,3)$ is a del Pezzo surface of degree 1 over k (see §2.2). Our main result is as follows.

Theorem 1.1. *Let $p \geq 5$ be a rational prime number such that $p \not\equiv 1 \bmod 12$. Let X be the del Pezzo surface of degree 1 over $\mathbb{Q}$ given by*

$$w^2 = z^3 + p^3x^6 + p^3y^6$$

in $\mathbb{P}_{\mathbb{Q}}(1,1,2,3)$. Then X is $\mathbb{Q}$-minimal and there is a Brauer-Manin obstruction to weak approximation on X. Moreover, the obstruction arises from a cyclic algebra class in $\operatorname{Br} X/\operatorname{Br}\mathbb{Q}$.

To obtain these examples, we begin with an explicit study of the geometry of *diagonal* del Pezzo surfaces of degree 1 over a perfect field k with $\operatorname{char} k \neq 2,3$. These are sextic surfaces of the form

$$w^2 = z^3 + Ax^6 + By^6 \tag{1}$$

in the weighted projective space $\mathbb{P}_k(1,1,2,3)$, where $A, B \in k^*$. The conditions A, $B \in k^*$ and $\operatorname{char} k \neq 2,3$, taken together, are equivalent to the smoothness of these surfaces.

Let R be a graded ring and let $I \subseteq R$ be a homogeneous ideal. Then $V(I) := \operatorname{Proj} R/I$. If $I = (f_1, \cdots f_n)$ we write $V(f_1, \ldots, f_n)$ instead of $V((f_1, \ldots, f_n))$. We start by finding an explicit description of generators for the geometric Picard group for the surfaces (1). More generally, we find explicit equations for all 240 exceptional curves on a del Pezzo surface of degree 1 over any perfect field.

Theorem 1.2. *Let X be a del Pezzo surface of degree 1 over a perfect field k, given as a smooth sextic hypersurface $V(f(x,y,z,w))$ in $\mathbb{P}_k(1,1,2,3)$. Let*

$$\Gamma = V(z - Q(x,y), w - C(x,y)) \subseteq \mathbb{P}_{\bar{k}}(1,1,2,3),$$

where $Q(x,y)$ and $C(x,y)$ are homogenous forms of degrees 2 and 3, respectively, in $\bar{k}[x,y]$. If Γ is a divisor on $X_{\bar{k}} := X \times_k \bar{k}$, then it is an exceptional curve of X. Conversely, every exceptional curve on X is a divisor of this form.

With explicit generators for $\operatorname{Pic} X_{\bar{k}}$ for a surface X of the form (1), we may compute the cohomology group $H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} X_{\bar{k}})$. We derive the following theorem, analogous to [KT04, Thm. 1].

Theorem 1.3. *Let k be a perfect field with $\operatorname{char} k \neq 2,3$. Let X be a minimal del Pezzo surface of degree 1 over k of the form (1). Then $H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic}(X_{\bar{k}}))$ is isomorphic to one of the following fourteen groups:*

$$\{1\}; \quad (\mathbb{Z}/2\mathbb{Z})^i, \quad i \in \{1,2,3,4,6,8\}; \quad (\mathbb{Z}/3\mathbb{Z})^j, \quad j \in \{1,2,3,4\};$$
$$(\mathbb{Z}/6\mathbb{Z})^k \quad k \in \{1,2\}; \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

Each group occurs for some field k. When $k = \mathbb{Q}$ only the following seven groups occur:

$$\{1\}, \quad \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$
$$\mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \quad \mathbb{Z}/6\mathbb{Z}.$$

If, furthermore, k is a number field, then we may compute the group $\operatorname{Br} X/\operatorname{Br} k$, of arithmetic interest, via the isomorphism

$$\operatorname{Br} X/\operatorname{Br} k \xrightarrow{\sim} H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} X_{\bar{k}}), \tag{2}$$

obtained from the Hochschild-Serre spectral sequence (see, for example, [CTS77, Lemme 15]).

To prove a statement like Theorem 1.1, we have to identify elements of $\operatorname{Br} X/\operatorname{Br} k$ explicitly. Given a cohomology class in $H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} X_{\bar{k}})$, it can be difficult to identify the corresponding element in $\operatorname{Br} X/\operatorname{Br} k$ guaranteed by the isomorphism (2). Hence, in §3 we present a simple strategy to search for cohomology classes in $H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic} X_{\bar{k}})$ which correspond to cyclic algebras in the image of the natural map

$$\operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k,$$

where X is a locally soluble smooth geometrically integral variety over a number field k. We hope that Theorem 3.3 will be of use to others wishing to calculate Brauer-Manin obstructions to the Hasse principle and weak approximation via cyclic algebras on this wide class of varieties.

The paper is organized as follows. In §2 we review a few basic facts about del Pezzo surfaces and Brauer-Manin obstructions. In §3 we present a strategy to search for cyclic algebras in the image of the natural map $\operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k$, as explained above. In §4 we prove Theorem 1.2 and in §5 we use it to write down generators for the geometric Picard group on a surface X of the form (1). In §6 we compute the action of $\operatorname{Gal}(\bar{k}/k)$ on $\operatorname{Pic} X_{\bar{k}}$ and the possible groups $H^1(\operatorname{Gal}(\bar{k}/k), \operatorname{Pic}(X_{\bar{k}}))$. Finally we prove Theorem 1.1 in §7.

1.1. **Notation.** In addition to the notation introduced above, we use the following conventions. Throughout k denotes a perfect field and $\bar{k}$ is a fixed algebraic closure of k. From §5 onwards we assume $\operatorname{char} k \neq 2, 3$; in this case A and B denote elements of k^*; α and β are fixed sixth roots of A and B, respectively, in $\bar{k}$. Also, ζ denotes a primitive sixth root of unity in $\bar{k}$ and s a fixed cube root of 2 in $\bar{k}$.

If X and Y are S-schemes then $X_Y := X \times_S Y$. If $Y = \operatorname{Spec} R$ then we write X_R instead of $X_{\operatorname{Spec} R}$. For an integral scheme X over a field we write $k(X)$ for the function field of X. A surface X is a separated integral scheme of finite type over a field k of dimension 2. If X is a locally factorial projective surface, then there is an intersection pairing on the Picard group $(\,\cdot\,,\cdot\,)_X \colon \operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$. We omit the subscript on the pairing if no confusion can arise. For such an X, we will identify $\operatorname{Pic}(X)$ with the Weil divisor class group; in particular, we will use additive notation for the group law on $\operatorname{Pic}(X)$.

For a smooth variety X over a number field k, and a Galois extension L/k, we write $N_{L/k} \colon \operatorname{Div} X_L \to \operatorname{Div} X_k$ and $\overline{N}_{L/k} \colon \operatorname{Pic} X_L \to \operatorname{Pic} X_k$ for the usual norm maps, respectively.

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2. Background

We begin by reviewing some well known facts about del Pezzo surfaces over a field k. The basic references on the subject are [Man74], [Dem80] and [Kol96, III.3]. Unless otherwise stated, X denotes a del Pezzo surface over a field k of degree d such that $X_{\bar{k}} \not\cong \mathbb{P}^1_{\bar{k}} \times \mathbb{P}^1_{\bar{k}}$.

2.1. **Picard groups.** Recall an exceptional curve on X is an irreducible curve C on $X_{\bar{k}}$ such that $(C, C) = (C, K_X) = -1$. When $d = 1$, X contains 240 exceptional curves. The group $\operatorname{Pic} X_{\bar{k}}$ is isomorphic to $\mathbb{Z}^{10-d}$; it is generated by the classes of exceptional curves. A possible basis for $\operatorname{Pic} X_{\bar{k}}$ is $\{e_1, \dots, e_{9-d}, \ell\}$, where each e_i is the class of an exceptional curve, and

$$(e_i, e_j) = -\delta_{ij}, \quad (e_i, \ell) = 0, \quad (\ell, \ell) = 1.$$

Under this basis, the anticanonical class is given by $-K_X = 3\ell - \sum e_i$.

2.2. **Anticanonical models.** If X is a del Pezzo surface then $X \cong \operatorname{Proj} \bigoplus_{m \geq 0} H^0(X, -mK_X)$ [Kol96, Theorem III.3.5]. The latter scheme is known as the anticanonical model of X. When $d = 1$ the anticanonical model is a smooth sextic hypersurface $V(f(x, y, z, w))$ in $\mathbb{P}_k(1, 1, 2, 3)$. Any smooth sextic hypersurface in $\mathbb{P}_k(1, 1, 2, 3)$ is a del Pezzo surface of degree 1. In this case $\{x, y\}$ is a basis for $H^0(X, -K_X)$ and $\{x^2, xy, y^2, z\}$ is a basis for $H^0(X, -2K_X)$.

2.3. **The Bertini involution.** Let X be a del Pezzo surface of degree 1 given as a smooth sextic $V(f)$ in $\mathbb{P}_k(1, 1, 2, 3)$. Write $f(x, y, z, w) = w^2 - aw - b$, where $a, b \in k[x, y, z]$ have degrees 3 and 6, respectively. If $\operatorname{char} k \neq 2$, then we may (and do) assume that $a \equiv 0$ by making the change of variables $w \mapsto w + a/2$. The map

$$\mathbb{P}_k(1, 1, 2, 3) \to \mathbb{P}_k(1, 1, 2, 3), \quad [x : y : z : w] \mapsto [x : y : z : -w + a]$$

restricts to an automorphism of X called the Bertini involution; see [Dem80, p. 68].

2.4. **Galois action on the Picard Group.** In this section X is a smooth, projective, geometrically rational surface over a number field k. Let K be the smallest subfield of $\bar{k}$ over which all exceptional curves of X are defined. We say K is the splitting field of X. The natural action of $\operatorname{Gal}(\bar{k}/k)$ on $\operatorname{Pic} X_{\bar{k}} \cong \operatorname{Pic} X_K$ factors through the quotient $\operatorname{Gal}(K/k)$, giving a map

$$\phi_X \colon \operatorname{Gal}(K/k) \to \operatorname{Aut}(\operatorname{Pic} X_K). \tag{3}$$

If we have equations for an exceptional curve C of X, then an element $\sigma \in \operatorname{Gal}(K/k)$ acts on C by applying σ to each coefficient. The curve ${}^{\sigma}C$ is itself an exceptional curve of X.

If, furthermore, X is a del Pezzo surface of degree 1. then the image of ϕ_X is isomorphic to a subgroup of the Weyl group $W(E_8)$ (which is a finite group of order $696, 729, 600$). To keep computations reasonable when searching for counterexamples to weak approximation, we work with surfaces X for which $\operatorname{im} \phi_X$ is small. On the other hand, the image cannot be too small: for example, if $\operatorname{im} \phi_X = \{1\}$, then X is k-birational to $\mathbb{P}^2_k$, so it satisfies weak approximation.

2.5. **Minimal surfaces.** There are examples of del Pezzo surfaces of degrees 2, 3 and 4 with a Zariski dense set of rational points for which weak approximation does not hold (cf. [KT07], [SD62] and [CTSSD87, 15.5], respectively). These examples can be used to construct nonminimal del Pezzo surfaces of degree 1 that do not satisfy weak approximation. To avoid such examples, we will insist that our surfaces be k-minimal.

Definition 2.1. We say X is k-minimal if there is no $\operatorname{Gal}(\bar{k}/k)$-stable set S of exceptional curves such that $(s_i, s_j) = -\delta_{ij}$ for every $s_i, s_j \in S$.

Del Pezzo surfaces X with $\operatorname{Pic} X \cong \mathbb{Z}$ are minimal. The converse is true if $d \notin \{1, 2, 4\}$; see [Man74, Rem. 28.1.1].

2.6. **Brauer-Manin obstructions.** We refer the reader to [Sko01] for a thorough treatment of the material in this section. If X is a smooth projective geometrically integral variety over a number field k, then the natural inclusion $X(\mathbb{A}_k) \subseteq \prod_{v\in\Omega_k} X(k_v)$ is a bijection. Let $\operatorname{Br} X$ be the group of equivalence classes of Azumaya algebras on X, and let $\operatorname{inv}_v \colon \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$ be the local invariant map. By class field theory there is a constraint

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} := \Big\{ (x_v)_v \in X(\mathbb{A}_k) \mid \sum_v \operatorname{inv}_v(\mathscr{A}(x_v)) = 0 \text{ for every } \mathscr{A} \in \operatorname{Br} X \Big\},$$

where $\mathscr{A}(x_v) := \mathscr{A}_{x_v} \otimes_{\mathscr{O}_{X,x_v}} k_v$. In fact, the closure $\overline{X(k)}$ of $X(k)$ in $X(\mathbb{A}_k)$ lies inside the set $X(\mathbb{A}_k)^{\mathrm{Br}}$. We say there is a Brauer-Manin obstruction to the Hasse principle (resp. weak approximation) if $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ but $X(\mathbb{A}_k) \neq \emptyset$ (resp. if $X(\mathbb{A}_k)^{\mathrm{Br}} \neq X(\mathbb{A}_k)$). We remark that to compute $X(\mathbb{A}_k)^{\mathrm{Br}}$ it suffices to consider a set of representatives of $\operatorname{Br} X/\operatorname{Br} k$. We also note that when $X(\mathbb{A}_k) \neq \emptyset$ the natural map $\operatorname{Br} k \to \operatorname{Br} X$ is an injection.

For X as above, we have $\operatorname{Br} X \cong H^2_{\text{ét}}(X, \mathbb{G}_m)$. This allows us to think of the Brauer group as a contravariant functor with values in the category of abelian groups.

3. Finding cyclic algebras in $\operatorname{Br} X$

When X is a regular, integral, quasi-compact scheme the natural map $\operatorname{Br} X \to \operatorname{Br} k(X)$ is injective (see [Mil80, III.2.22]). There are certain elements of $\operatorname{Br} k(X)$ whose local invariants are easy to compute. They are the cyclic algebras.

3.1. **Review of cyclic algebras.** Let L/k be a finite cyclic extension of fields of degree n. Fix a generator σ of $\operatorname{Gal}(L/k)$. Let $L[x]_\sigma$ be the "twisted" polynomial ring, where $\ell x = x^\sigma \ell$ for all $\ell \in L$. Given $b \in k^*$ we construct the central simple k-algebra $L[x]_\sigma/(x^n - b)$. This algebra is usually denoted $(L/k, b)$: it depends on the choice of σ, though the notation does not show this.

If X is a geometrically integral k-variety, then the cyclic algebra $(k(X_L)/k(X), f)$ is also denoted $(L/k, f)$; this should not cause confusion because $\operatorname{Gal}(k(X_L)/k(X)) \cong \operatorname{Gal}(L/k)$.

The following is a criterion for testing whether or not a cyclic algebra is in the image of the map $\operatorname{Br} X \to \operatorname{Br} k(X)$. For a proof, see [Cor05, Prop. 2.2.3] or [Bri02, Prop. 4.17]. See §1.1 for our conventions on the norm maps $N_{L/k}$ and $\overline{N}_{L/k}$.

Proposition 3.1. *Let X be a smooth, geometrically integral variety over a number field k. Let L/k a finite cyclic extension and $f \in k(X)^*$. Then the cyclic algebra $(L/k, f)$ is in the image of the natural map $\operatorname{Br}(X) \to \operatorname{Br} k(X)$ if and only if $(f) = N_{L/k}(D)$, for some $D \in \operatorname{Div} X_L$. If $X(k_v) \neq \emptyset$ for all $v \in \Omega_k$ then $(L/k, f)$ comes from $\operatorname{Br} k$ if and only if we can take D to be principal.* □

3.2. **Cyclic Azumaya algebras.** Let X be a smooth geometrically integral variety over a number field k. Assume that $X(k_v) \neq \emptyset$ for all $v \in \Omega_k$. By functoriality of the Brauer group we have maps $\operatorname{Br} k \to \operatorname{Br} X \to \operatorname{Br} k(X)$, where the first map is an injection (see §2.6).

Let L/k be a cyclic extension. Define the set

$$\operatorname{Br}_{\mathrm{cyc}}(X, L) := \left\{ \begin{matrix} \text{classes } [(L/k, f)] \text{ in the image of the} \\ \text{map } \operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k \end{matrix} \right\}$$

Lemma 3.2. *Viewing $\Delta := 1 - \sigma$ as an endomorphism of $\operatorname{Div} X_L$, we have $\ker N_{L/k} = \operatorname{im} \Delta$.*

Proof. By Tate cohomology we know that $H^1(\mathrm{Gal}(L/k), \operatorname{Div} X_L) \cong \ker N_{L/k}/\operatorname{im}\Delta$. On the other hand, this cohomology group is trivial: $\operatorname{Div} X_L$ is a permutation module, so the result follows from Shapiro's Lemma. ☐

The ideas behind the following theorem can be found in [Bri02, §4.3.2, esp. Lem. 4.18].

Theorem 3.3. *Let X be a k-variety as above. Let H be an open normal subgroup of $G := \mathrm{Gal}(\bar{k}/k)$, such that G/H is cyclic, generated by σ. Let L be the fixed field of H. The map*

$$\psi\colon \ker \overline{N}_{L/k}/\operatorname{im}\Delta \to \mathrm{Br}_{\mathrm{cyc}}(X, L) \qquad [D] \mapsto [(L/k, f)],$$

where $f \in k(X)^*$ is any function such that $N_{L/k}(D) = (f)$, is a group isomorphism.

Proof. First we check ψ is well-defined by showing that

(1) the class $[(L/k, f)]$ is independent of the choice of f: if $N_{L/k}(D) = (f) = (g)$, then $g = af$ for some $a \in k^*$. Since $(L/k, a) \in \operatorname{Br} k$, we obtain $[(L/k, f)] = [(L/k, g)]$.

(2) if D and D' are linearly equivalent divisors in $\ker \overline{N}_{L/k}$, with $N_{L/k}(D) = (f)$ and $N_{L/k}(D') = (f')$, then $[(L/k, f)] = [(L/k, f')]$: equivalently, by Proposition 3.1 we need (f/f') to be the norm of a principal divisor. Say $D = D' + (h)$. Then $(f/f') = N_{L/k}((h))$.

(3) an element in $\operatorname{im}\Delta$ maps to zero: this is trivial.

If $N_{L/k}(D) = (f)$ and $N_{L/k}(D') = (g)$ then

$$(4)\quad \psi([D] + [D']) = \psi([D + D']) = [(L/k, fg)] = [(L/k, f)] + [(L/k, g)] = \psi([D]) + \psi([D']),$$

so ψ is a homomorphism. The map ψ is injective: if $\psi([D]) = [(L/k, f)]$ is 0 in $\operatorname{Br} k(X)/\operatorname{Br} k$, then by Proposition 3.1 there exists an $h \in k(X_L)^*$ such that $(f) = N_{L/k}((h))$. Hence $D - (h) \in \ker N_{L/k} = \operatorname{im}\Delta$ (see Lemma 3.2). Surjectivity also follows from Proposition 3.1: given a class $[(L/k, f)]$, take any divisor D such that $N_{L/k}(D) = (f)$; then $\psi([D]) = [(L/k, f)]$. ☐

3.3. **Cyclic algebras on rational surfaces.** Let X be a smooth, projective, geometrically integral rational surface over a number field k, and let K be the splitting field of X. Assume that $X(\mathbb{A}_k) \neq \emptyset$. The inflation map

$$(5)\qquad H^1(\mathrm{Gal}(K/k), \operatorname{Pic} X_K) \to H^1(\mathrm{Gal}(\bar{k}/k), \operatorname{Pic} X_{\bar{k}})$$

is an isomorphism, because the cokernel maps into the first cohomology group of a free $\mathbb{Z}$-module with trivial action by a profinite group, so it is trivial. By (2) it follows that

$$(6)\qquad \operatorname{Br} X/\operatorname{Br} k \cong H^1(\mathrm{Gal}(K/k), \operatorname{Pic} X_K).$$

Let $G = \mathrm{Gal}(K/k)$ and suppose that H is a normal subgroup of G such that G/H is cyclic. Let L be the fixed field of H. Since $X(\mathbb{A}_k) \neq \emptyset$, it follows from the Hochschild-Serre spectral sequence and (2) that

$$(7)\qquad (\operatorname{Pic} X_K)^H \cong \operatorname{Pic} X_L.$$

We obtain an injection

$$(8)\qquad H^1(\mathrm{Gal}(L/k), \operatorname{Pic} X_L) \xrightarrow{\inf} H^1(G, \operatorname{Pic} X_K) \cong \operatorname{Br} X/\operatorname{Br} k$$

On the other hand, by Tate cohomology we know that

$$H^1(\mathrm{Gal}(L/k), \operatorname{Pic} X_L) \cong \ker \overline{N}_{L/k}/\operatorname{im}\Delta.$$

We can use Theorem 3.3 to write down cyclic algebras $(L/k, f)$ in the image of the injection $\operatorname{Br} X/\operatorname{Br} k \to \operatorname{Br} k(X)/\operatorname{Br} k$. Since G is finite, we may search through its subgroup lattice to find subgroups H as above, and hence write down all the cyclic algebras in $\operatorname{Br} X/\operatorname{Br} k$.

It will be important for us to determine the functions f above explicitly; to this end, we must make the isomorphism (7) explicit. This is explained in the Appendix.

Remark 3.4. Finding Brauer-Manin obstructions to the *Hasse principle* on del Pezzo surfaces of degree greater than 1 may require the injection (8) to be an isomorphism. (This will be the case, for example, if $H^1(H, \operatorname{Pic} X_K) = 0$). We may need representative Azumaya algebras for *every* class in $\operatorname{Br} X/\operatorname{Br} k$ to detect a Brauer-Manin obstruction (for example, see [Cor07, 9.4]). Obstructions to weak approximation only require one Azumaya algebra.

Remark 3.5. Let X be a diagonal del Pezzo surface of degree 1 over $\mathbb{Q}$ such that the order of $\operatorname{Br} X/\operatorname{Br} \mathbb{Q}$ is divisible by 3. Let K be the splitting field of X. Then an exhaustive computer search reveals that there does not exist a normal subgroup H of $G := \operatorname{Gal}(K/\mathbb{Q})$ such that $|G/H|$ is divisible by 3. This means that any counterexamples to weak approximation over $\mathbb{Q}$ we find using the above strategy will always arise from 2-torsion Azumaya algebras.

Remark 3.6. Not all Brauer-Manin obstructions on del Pezzo surfaces arise from cyclic algebras: for example, see [KT04, Ex. 8].

4. Exceptional curves on del Pezzo surfaces of degree 1

In this section we assume k is algebraically closed. Let

$$\Gamma := V(z - Q(x,y), w - C(x,y)) \subseteq \mathbb{P}_k(1,1,2,3),$$

where $Q(x,y)$ and $C(x,y)$ are homogenous forms of degrees 2 and 3, respectively, in $k[x,y]$. Define Γ' as the image of Γ under the Bertini involution (see §2.3). Note that $\Gamma \neq \Gamma'$.

Lemma 4.1. *Let X be a del Pezzo surface of degree 1, given as a sextic hypersurface in $\mathbb{P}_k(1,1,2,3)$. If Γ is a divisor on X then so is Γ'; in this case $(\Gamma, \Gamma')_X = 3$.*

Proof. It is clear that if Γ is a divisor on X then so is Γ'. Assume first that $\operatorname{char} k \neq 2$. Note $(\Gamma, \Gamma')_X$ is equal to the degree of the scheme $\Gamma \cap \Gamma'$, whose defining ideal is

$$(z - Q(x,y), w - C(x,y), w + C(x,y)) = (z - Q(x,y), w, C(x,y)).$$

We compute

$$\deg(\operatorname{Proj} k[x,y,z,w]/(z - Q, w, C)) = \deg(\operatorname{Proj} k[x,y]/(C)) = 3.$$

When $\operatorname{char} k = 2$, the ideal of $\Gamma \cap \Gamma'$ is $(z + Q(x,y), w + C(x,y), a)$. A calculation similar to the one above shows that $(\Gamma, \Gamma')_X = 3$. $\square$

4.1. **The bianticanonical map.** Let X be a del Pezzo surface of degree 1 over k. The map

$$\phi_2 \colon X \to \mathbb{P}(H^0(X, -2K_X)^*) = \mathbb{P}^3_k.$$

is known as the bianticanonical map. If $X = V(f(x,y,z,w)) \subseteq \mathbb{P}_k(1,1,2,3)$, then the basis elements x^2, xy, y^2, z for $H^0(X, -2K_X)$ are homogeneous coordinates for ϕ_2 (see §2.2). Let $T_0, \ldots, T_3$ be coordinates for $\mathbb{P}^3_k$. The map ϕ_2 is 2-to-1 onto the quadric cone $\mathcal{Q} = V(T_0T_2 - T_1^2)$. This cone is in turn isomorphic to the space $\mathbb{P}_k(1,1,2)$ via the map

$$j \colon \mathbb{P}_k(1,1,2) \to \mathcal{Q}, \quad [x : y : z] \mapsto [x^2 : xy : y^2 : z].$$

The composition $j^{-1}\circ\phi_2\colon X \to \mathbb{P}_k(1,1,2)$ is just the restriction to X of the natural projection $\mathbb{P}_k(1,1,2,3) \dashrightarrow \mathbb{P}_k(1,1,2)$. We fix the notation $\pi_2 := j^{-1}\circ\phi_2$ for future reference.

Lemma 4.2 ([CO99])**.** *Let V denote the vertex of the cone $\mathcal{Q}$, and let Γ be an exceptional curve on X. Then $\phi_2|_\Gamma\colon \Gamma \to \phi_2(\Gamma)$ is 1-to-1 and $\phi_2(\Gamma)$ is a smooth conic, the intersection of $\mathcal{Q}$ with a hyperplane H that misses V.* □

Remark 4.3. The image of the anticanonical point under ϕ_2 is $V \in \mathcal{Q}$. By Lemma 4.2, the anticanonical point does not lie on any exceptional curve of X.

4.2. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. We may assume k is algebraically closed, as the statement of the theorem is geometric. First, we show that any Γ as in the theorem is an exceptional curve by proving that $(\Gamma, K_X)_X = (\Gamma,\Gamma)_X = -1$. Note $V(x) \in |-K_X|$. Hence

$$(\Gamma, -K_X)_X = \deg(\operatorname{Proj} k[x,y,z,w]/(z-Q, w-C, x)) = \deg(\operatorname{Proj} k[y]) = 1.$$

Let $D = V(z - Q(x,y)) \subseteq \mathbb{P}_k(1,1,2)$. Since D is isomorphic under j to a hyperplane section of the cone $\mathcal{Q}$, we have $\pi_2^*(D) \in |-2K_X|$, so $(\pi_2^*(D), \pi_2^*(D))_X = 4$. Define Γ' as the image of Γ under the Bertini involution (see §2.3). By Lemma 4.1, Γ' is a divisor on X. Since $\Gamma + \Gamma' \subseteq \pi_2^*(D)$, the divisor $\pi_2^*(D)$ is reducible, and since $\deg \pi_2 = 2$, it must consist of two distinct components with multiplicity 1 (because $\Gamma \neq \Gamma'$), that is, $\Gamma + \Gamma' = \pi_2^*(D)$. The Bertini involution interchanges Γ and Γ', so we must have $(\Gamma, \Gamma+\Gamma')_X = (\Gamma', \Gamma+\Gamma')_X = 2$. Thus $(\Gamma,\Gamma)_X = -1$ if and only if $(\Gamma,\Gamma')_X = 3$, but this follows from Lemma 4.1. Hence Γ is an exceptional curve. As above, we can show that $(\Gamma', -K_X)_X = 1$, so Γ' is also an exceptional curve.

Now we prove the converse. Let Γ be an exceptional curve on X. By Lemma 4.2 we know $\phi_2(\Gamma)$ is a smooth conic. It is isomorphic under the map j to the curve $\pi_2(\Gamma)$ in $\mathbb{P}_k(1,1,2)$. The equation for the conic in $\mathbb{P}_k(1,1,2)$ can be written as $z = Q(x,y)$, where $Q(x,y)$ is homogenous of degree 2 in $k[x,y]$ (the coefficient of z is non-zero because $\phi_2(\Gamma)$ misses the vertex V of the cone $\mathcal{Q}$).

Let $D = V(z - Q(x,y)) \subseteq \mathbb{P}_k(1,1,2)$, as before. We have shown that $\Gamma \subseteq \pi_2^*(D)$. Since $\pi_2^*(D) \in |-2K_X|$ as above, we have $(\pi_2^*(D), \Gamma)_X = 2$. If $\pi_2^*(D) = m\Gamma$ for some $m \geq 1$ then

$$2 = (\pi_2^*(D), \Gamma)_X = m(\Gamma,\Gamma)_X = -m,$$

a contradiction. Hence $\pi_2^*(D)$ is reducible, and $\pi_2^*(D) = \Gamma + \Gamma_1$ for some irreducible divisor $\Gamma_1 \neq \Gamma$. Note that

$$(\Gamma_1,\Gamma_1)_X = (\pi_2^*(D) - \Gamma, \pi_2^*(D) - \Gamma)_X = (-2K_X - \Gamma, -2K_X - \Gamma)_X = -1,$$

and similarly $(\Gamma_1, -K_X)_X = 1$, so Γ_1 is an exceptional curve of X. We have

$$\pi_2^*(D) = V(f(x,y,z,w), z - Q(x,y)).$$

On the affine open subset where $x \neq 0$, the coordinate ring of $\pi_2^*(D)$ is

$$k[y,z,w]/(f(1,y,z,w), z - Q(1,y)) \cong k[y,w]/(f(1,y,Q(1,y),w)).$$

Since $\pi_2^*(D)$ is reducible, the polynomial $f(1,y,Q(1,y),w)$ must factor, and degree considerations force a factorization of the following form:

$$(w - C(1,y))(w - C'(1,y)),$$

where $C(x,y)$ and $C'(x,y)$ are homogeneous forms of degree 3. Hence Γ has the form we claimed. $\square$

Remark 4.4. The divisor Γ_1 in the proof above is the image of Γ under the Bertini involution.

Remark 4.5. We have used several ideas from the proof of [CO99, Key-lemma 2.7] to prove Theorem 1.2. The theorem can also be deduced from the work of Shioda on rational elliptic surfaces $S \to \mathbb{P}^1$ (see [Shi90, Thm. 10.10]). Shioda shows that rational elliptic surfaces have at most 240 sections $\mathbb{P}^1 \to S$ of a particular form, whose description bares a striking resemblance to the divisors of the form Γ above. A rational elliptic surface (over an algebraically closed field) with exactly 240 of these special sections corresponds to the blow up of a del Pezzo surface X of degree 1 with center at the anticanonical point; the special sections of the elliptic surface are in one to one correspondence with the exceptional curves of X. Under this correspondence, Shioda's explicit description of the 240 sections becomes the explicit description of the exceptional curves of Theorem 1.2. Cragnolini and Oliverio have a somewhat different description of the exceptional curves on a del Pezzo surface of degree 1 [CO99, Key-lemma 2.7] (see also [Dem80, p. 68]).

Remark 4.6. Suppose k is not algebraically closed. The Bertini involution interchanges Γ and Γ'; since it is defined over k we conclude that

$$^{\sigma}(\Gamma') = (^{\sigma}\Gamma)' \quad \text{for all } \sigma \in \operatorname{Gal}(\bar{k}/k).$$

We will therefore use the unambiguous notation $^{\sigma}\Gamma'$ for this divisor.

5. Exceptional curves on diagonal surfaces

We begin by studying the particular surface Y given by the sextic $w^2 = z^3 + x^6 + y^6$ in $\mathbb{P}_k(1,1,2,3)$. Suppose first that $k = \overline{\mathbb{Q}}$. By Theorem 1.2, the exceptional curves on Y are given as $V(w - C(x,y), z - Q(x,y))$, where

$$C(x,y)^2 = Q(x,y)^3 + x^6 + y^6.$$

Using Gröbner bases in `Magma` to solve for the coefficients of Q and C, we find 240 exceptional curves, all defined over $\mathbb{Q}(\sqrt[3]{2}, \zeta)$.

If k is algebraically closed of characteristic 0 the equations for the exceptional curves we calculated over $\overline{\mathbb{Q}}$ give exceptional curves over k via an embedding $\iota\colon \overline{\mathbb{Q}} \hookrightarrow k$.

Now suppose k is algebraically closed of characteristic $p > 3$. Let $W(k)$ be the ring of Witt vectors of k, and let $F(k)$ be its field of fractions. Let $\mathcal{X}$ be the del Pezzo surface over $W(k)$ given by the equation $w^2 = z^3 + x^6 + y^6$ in $\mathbb{P}_{W(k)}(1,1,2,3)$. The generic fiber of $\mathcal{X}$ is a del Pezzo surface over $F(k)$. We may write down its 240 exceptional curves as above: even though $F(k)$ is not algebraically closed, we may embed $\mathbb{Q}(\sqrt[3]{2}, \zeta)$ in it, and this is enough to write down equations for all the exceptional curves.

The usual specialization map $\theta\colon \operatorname{Pic} \mathcal{X}_{F(k)} \to \operatorname{Pic} \mathcal{X}_k$ is a homomorphism (see [Ful98, §20.3]). In other words, θ preserves the intersection pairings on $\operatorname{Pic} \mathcal{X}_{F(k)}$ and $\operatorname{Pic} \mathcal{X}_k$; it is injective because the pairing on $\operatorname{Pic} \mathcal{X}_{F(k)}$ is nondegenerate. A standard computation shows that $\theta(K_{\mathcal{X}_{F(k)}}) = K_{\mathcal{X}_k}$ (see [Ful98, §20.3.1]). Hence θ maps exceptional curves to exceptional curves. The injectivity of θ then shows that the 240 exceptional curves on $\mathcal{X}_{F(k)}$ specialize to 240 *distinct* exceptional curves.

Let us drop the assumption that k is algebraically closed. We turn to the general diagonal surface X over k, given by $w^2 = z^3 + Ax^6 + By^6$. Fix a sixth root α of A and a sixth root β of B in $\bar{k}$. If $\Gamma = V(z - Q(x, y), w - C(x, y))$ is an exceptional curve on $w^2 = z^3 + x^6 + y^6$, then $V(z - Q(\alpha x, \beta y), w - C(\alpha x, \beta y))$ is an exceptional curve on X, and vice versa. We deduce that the splitting field of X is contained in $k(\zeta, \sqrt[3]{2}, \alpha, \beta)$.

Proposition 5.1. *Let k be a perfect field with* $\operatorname{char} p \neq 2, 3$*. Let X be the del Pezzo surface of degree 1 over k given by*

$$w^2 = z^3 + Ax^6 + By^6,$$

in $\mathbb{P}_k(1, 1, 2, 3)$. Then the splitting field of X is $K := k(\zeta, \sqrt[3]{2}, \alpha, \beta)$.

Proof. Let L denote the splitting field of X. The above discussion shows that $L \subseteq K$. Let $s = \sqrt[3]{2}$. By Theorem 1.2, the subschemes of $\mathbb{P}_k(1, 1, 2, 3)$ given by

$$\begin{aligned}
&V(z - s\alpha\beta xy, w - \alpha^3x^3 - \beta^3y^3),\\
&V(z + s\zeta\alpha\beta xy, w - \alpha^3x^3 - \beta^3y^3),\\
&V(z + \alpha^2x^2 - s^2\zeta\beta^2y^2, w - s(\zeta + 1)\alpha^2\beta x^2y + (2\zeta - 1)\beta^3y^3) \text{ and}\\
&V(z - s^2\zeta\alpha^2x^2 + \beta^2y^2, w - (2\zeta - 1)\alpha^3x^3 + s(\zeta + 1)\alpha\beta^2xy^2)
\end{aligned}$$

are exceptional curves on X. By definition of L, we find that

$$S := \{s\alpha\beta, s\zeta\alpha\beta, s(\zeta + 1)\alpha\beta^2, s(\zeta + 1)\alpha^2\beta\} \subseteq L.$$

Taking the quotient of the second element of S by the first shows that $\zeta \in L$. We also have $s(\zeta+1)\alpha\beta \in L$, which shows $s(\zeta+1)\alpha\beta^2/s(\zeta+1)\alpha\beta = \beta \in L$. Similarly $s(\zeta+1)\alpha^2\beta/s(\zeta+1)\alpha\beta = \alpha \in L$. Finally, we deduce that $s \in L$. This shows $K \subseteq L$. $\square$

To end our discussion on exceptional curves on diagonal surfaces, we give generators for $\operatorname{Pic} X_{\bar{k}}$ in terms of these curves. Consider the following exceptional curves on X:

$$\begin{aligned}
\Gamma_1 = V(&z + \alpha^2x^2, w - \beta^3y^3),\\
\Gamma_2 = V(&z - (-\zeta + 1)\alpha^2x^2, w + \beta^3y^3),\\
\Gamma_3 = V(&z - \zeta\alpha^2x^2 + s^2\beta^2y^2, w - (s\zeta - 2s)\alpha^2\beta x^2y - (-2\zeta + 1)\beta^3y^3),\\
\Gamma_4 = V(&z + 2\zeta\alpha^2x^2 - (2s\zeta - s)\alpha\beta xy - (-s^2\zeta + s^2)\beta^2y^2,\\
&w - 3\alpha^3x^3 - (-2s\zeta - 2s)\alpha^2\beta x^2y - 3s^2\zeta\alpha\beta^2xy^2 - (-2\zeta + 1)\beta^3y^3),\\
\Gamma_5 = V(&z + 2\zeta\alpha^2x^2 - (s\zeta - 2s)\alpha\beta xy - s^2\zeta\beta^2y^2\\
&w + 3\alpha^3x^3 - (4s\zeta - 2s)\alpha^2\beta x^2y - 3s^2\alpha\beta^2xy^2 - (-2\zeta + 1)\beta^3y^3),\\
\Gamma_6 = V(&z - (-s^2\zeta + s^2 - 2s + 2\zeta)\alpha^2x^2 - (2s^2\zeta - 2s^2 + 3s - 4\zeta)\alpha\beta xy - (-s^2\zeta + s^2 - 2s + 2\zeta)\beta^2y^2,\\
&w - (2s^2\zeta - 4s^2 + 2s\zeta + 2s - 6\zeta + 3)\alpha^3x^3 - (-5s^2\zeta + 10s^2 - 6s\zeta - 6s + 16\zeta - 8)\alpha^2\beta x^2y\\
&\quad - (5s^2\zeta - 10s^2 + 6s\zeta + 6s - 16\zeta + 8)\alpha\beta^2xy^2 - (-2s^2\zeta + 4s^2 - 2s\zeta - 2s + 6\zeta - 3)\beta^3y^3),\\
\Gamma_7 = V(&z - (-s^2 - 2s\zeta + 2s + 2\zeta)\alpha^2x^2 - (-2s^2\zeta + 3s + 4\zeta - 4)\alpha\beta xy - (-s^2\zeta + s^2 + 2s\zeta - 2)\beta^2y^2,\\
&w - (2s^2\zeta + 2s^2 + 2s\zeta - 4s - 6\zeta + 3)\alpha^3x^3 - (10s^2\zeta - 5s^2 - 6s\zeta - 6s - 8\zeta + 16)\alpha^2\beta x^2y\\
&\quad - (5s^2\zeta - 10s^2 - 12s\zeta + 6s + 8\zeta + 8)\alpha\beta xy^2 - (-2s^2\zeta - 2s^2 - 2s\zeta + 4s + 6\zeta - 3)\beta^3y^3),\\
\Gamma_8 = V(&z - (s^2\zeta + 2s\zeta + 2\zeta)\alpha^2x^2 - (2s^2 + 3s + 4)\alpha\beta xy - (-s^2\zeta + s^2 - 2s\zeta + 2s - 2\zeta + 2)\beta^2y^2,\\
&w - (-4s^2\zeta + 2s^2 - 4s\zeta + 2s - 6\zeta + 3)\alpha^3x^3 - (-5s^2\zeta - 5s^2 - 6s\zeta - 6s - 8\zeta - 8)\alpha^2\beta x^2y\\
&\quad - (5s^2\zeta - 10s^2 + 6s\zeta - 12s + 8\zeta - 16)\alpha\beta^2xy^2 - (4s^2\zeta - 2s^2 + 4s\zeta - 2s + 6\zeta - 3)\beta^3y^3).
\end{aligned}$$

	σ	τ	ι_A	ι_B
$\sqrt[3]{2}$	$-\zeta\sqrt[3]{2}$	$\sqrt[3]{2}$	$\sqrt[3]{2}$	$\sqrt[3]{2}$
ζ	ζ	ζ^{-1}	ζ	ζ
α	α	α	$\zeta\alpha$	α
β	β	β	β	$\zeta\beta$

TABLE 1. Action of the generators of $\mathrm{Gal}(K/k)$, assuming $\sqrt[3]{2}, \zeta \notin k$.

A calculation shows that the above exceptional curves are all skew, that is, $(\Gamma_i, \Gamma_j)_X = 0$ for $i \neq j$. We will also need the exceptional curve

$$\Gamma_9 = V(z - st\alpha\beta xy, w - \alpha^3 x^3 + \beta^3 y^3).$$

The curve Γ_9 intersects Γ_1 and Γ_2 at exactly one point and is skew to all the other Γ_i.

Proposition 5.2. *Let X be the del Pezzo surface over k defined by*

$$w^2 = z^3 + Ax^6 + By^6,$$

in $\mathbb{P}_k(1,1,2,3)$. Then $\operatorname{Pic} X_{\bar{k}} = \operatorname{Pic} X_K$ is the free abelian group with the classes of Γ_i for $1 \leq i \leq 8$ and $\Gamma_9 + \Gamma_1 + \Gamma_2$ as a basis.

Proof. By Proposition 5.1 we know K is the splitting field of X. The classes of Γ_i for $1 \leq i \leq 8$ and $\Gamma_9 + \Gamma_1 + \Gamma_2$ generate a *unimodular* sublattice of $\operatorname{Pic} X_K$ of rank 9. Hence they span the whole lattice. $\square$

6. Galois action on $\operatorname{Pic} X_K$

Suppose $\sqrt[3]{2}, \zeta \notin k$ and let X be a generic surface of the form (1). Let $K = k(\zeta, \sqrt[3]{2}, \alpha, \beta)$, as above. The action of $\mathrm{Gal}(\bar{k}/k)$ on $\operatorname{Pic} X_{\bar{k}}$ factors through the finite quotient $\mathrm{Gal}(K/k)$, which acts on the coefficients of the equations defining generators of $\operatorname{Pic} X_K$ (cf. §2.4). The group $\mathrm{Gal}(K/k)$ has 4 generators, which we will denote $\sigma, \tau, \iota_A, \iota_B$, whose action on the elements $\zeta, \sqrt[3]{2}, \alpha$ and β is recorded in Table 1. If $\sqrt[3]{2} \in k$ (resp. $\zeta \in k$), then we do not need the generator σ (resp. τ).

Using the basis for $\operatorname{Pic} X_K$ of Proposition 5.2 we can write σ, τ, ι_A and ι_B as 9×9 matrices with integer entries. This 9-dimensional faithful representation is useful because the action of $\mathrm{Gal}(K/k)$ on $\operatorname{Pic} X_K$ becomes right matrix multiplication on the space of row vectors $\mathbb{Z}^9$.

Proof of Theorem 1.3. Assume first that $\sqrt[3]{2}, \zeta \notin k$. Then $G_0 := \langle \sigma, \tau, \iota_A, \iota_B \rangle \subseteq GL_9(\mathbb{Z})$ is isomorphic to the generic image of $\mathrm{Gal}(\bar{k}/k)$ in $\mathrm{Aut}(\operatorname{Pic} X_{\bar{k}})$ for a diagonal del Pezzo surface of degree 1. For a particular surface, a choice of sixth roots α and β of A and B, respectively, and a sixth root of unity ζ gives a realization of $G := \mathrm{Gal}(K/k)$ as a subgroup of G_0, where $K = k(\zeta, \sqrt[3]{2}, \alpha, \beta)$.

We turn this idea around by focusing on the subgroup lattice of G_0. We use `Magma` to compute the first group cohomology (with coefficients in $\operatorname{Pic} X_K$) of subgroups in this lattice. We note there is no need to compute this cohomology group for every subgroup in the lattice. For example, any two subgroups of G_0 conjugate in $W(E_8)$ give rise to isomorphic cohomology groups. There are 448 conjugacy classes of subgroups of G_0 in $W(E_8)$.

We also note that in order for a subgroup $G \subseteq G_0$ to correspond to at least one diagonal del Pezzo surface of degree 1, it is necessary that the natural map $G \to G_0/\langle \iota_A, \iota_B \rangle$ be surjective because $k(\zeta, \sqrt[3]{2}) \subseteq K$. This cuts the number of conjugacy classes for which we need to compute group cohomology to 242.

Fix a subgroup $G \subseteq G_0$. For each exceptional curve Γ (given as a row vector in $\mathbb{Z}^9$, using Proposition 5.2) we may compute the orbit of Γ under the action of G. If there is a G-stable set of *skew* exceptional curves, then any surface X that has G for its image of $\mathrm{Gal}(\bar{k}/k)$ in $\mathrm{Aut}(\mathrm{Pic}\, X_{\bar{k}})$ is not minimal. Hence, we discard any such G. This way we get rid of 58 conjugacy classes of subgroups of G_0 and guarantee that surfaces we deal with in the rest of the paper are minimal.

The above reductions cut the number of candidate groups for G to 184. The results of our computations are summarized in Table 2. For each abstract group $\mathrm{Br}\, X/\,\mathrm{Br}\, k$ we list the number $C(G)$ of conjugacy classes of subgroups of G_0 that give the listed cohomology group. We also give an example of a subgroup $G \subseteq G_0$ that has the given cohomology group, and a pair of elements $A, B \in k^*$ such that the surface X of the form (1) realizes G as a Galois group acting on $\mathrm{Pic}\, X_{\bar{k}}$. This shows all the possible cohomology groups *do* occur.

If $\sqrt[3]{2} \in k$ yet $\zeta \notin k$ then we may repeat the above process starting with $G_0 = \langle \tau, \iota_A, \iota_B \rangle$. If $\zeta \in k$ yet $\sqrt[3]{2} \notin k$ then we use $G_0 = \langle \sigma, \iota_A, \iota_B \rangle$. Finally, if $\zeta, \sqrt[3]{2} \in k$ then we use $G_0 = \langle \iota_A, \iota_B \rangle$. The results in these three cases are summarized in Table 2. □

Looking through our computations we observe that

$$H^1(G_0, \mathrm{Pic}\, X_K) = 0,$$

regardless of whether the elements $\sqrt[3]{2}$ and ζ belong to k or not. This means that *generically there is no Brauer–Manin obstruction to weak approximation* on diagonal del Pezzo surfaces of degree 1 over a number field.

Remark 6.1. In [Cor07, Theorem 4.1] Corn determines the possible groups

$$\mathrm{Br}\, X/\,\mathrm{Br}\, k \cong H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Pic}\, X_{\bar{k}})$$

for all del Pezzo surfaces X over a number field k. In particular, Corn shows the only primes that divide the order of this group are 2, 3 and 5, and the latter can only occur when X is of degree 1. Unfortunately, *diagonal* surfaces of degree 1 cannot be used to give examples of 5-torsion in $\mathrm{Br}\, X/\,\mathrm{Br}\, k$. This follows either from Theorem 1.3 or, more easily, from the isomorphism (5): the group $H^1(\mathrm{Gal}(K/k), \mathrm{Pic}\, X_K)$ is annihilated by $[K : k]$, which divides 216, by Proposition 5.1.

7. Counterexamples to Weak Approximation

7.1. **A warm-up example.** We begin with an example over $k = \mathbb{Q}(\zeta)$ for which we do not need to use the descent procedure described in the Appendix, and for $\mathrm{Gal}(K/k)$ is small. The presence of an obstruction to weak approximation on it cannot be explained by a conic bundle structure (see Remark 7.2).

Proposition 7.1. *Let X be the del Pezzo surface of degree 1 over $k = \mathbb{Q}(\zeta)$ given by*

$$w^2 = z^3 + 16x^6 + 16y^6$$

in $\mathbb{P}_k(1, 1, 2, 3)$. Then X is k-minimal and there is a Brauer-Manin obstruction to weak approximation on X. Moreover, the obstruction arises from a cyclic algebra class in $\mathrm{Br}\, X/\,\mathrm{Br}\, k$.

	$\operatorname{Br} X / \operatorname{Br} k$	$C(G)$	Example of G	A, B	Restrictions
$\sqrt[3]{2} \notin k$, $\zeta \notin k$	$\{1\}$	65	$\langle \sigma\iota_B^4, \tau, \iota_A^2 \rangle$	$a^2c^6, \pm 4d^6$	$a \notin \langle 2, k^{*3} \rangle$
	$\mathbb{Z}/2\mathbb{Z}$	18	$\langle \sigma\iota_A, \tau, \iota_B^3 \rangle$	$4a^3c^6, b^3d^6$	$a, b \notin \langle 2, -3, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^2$	9	$\langle \sigma, \tau, \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin \langle 2, -3, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^3$	4	$\langle \sigma\iota_A^2, \tau, \iota_A^3\iota_B^3 \rangle$	$16a^3c^6, a^3d^6$	$a \notin \langle 2, -3, k^{*2} \rangle$
	$\mathbb{Z}/3\mathbb{Z}$	56	$\langle \sigma\iota_A\iota_B^2, \iota_A^3, \tau \rangle$	$4a^3c^6, \pm 16d^6$	$a \notin \langle -3, k^{*2} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^2$	26	$\langle \tau, \sigma\iota_A^2\iota_B^2 \rangle$	ac^6, ad^6	$a \in \pm 16k^{*6}$
	$\mathbb{Z}/6\mathbb{Z}$	6	$\langle \sigma\iota_A, \iota_A^3, \tau\iota_B \rangle$	$4a^3c^6, -3d^6$	$a \notin \langle 3, k^{*2} \rangle$
$\sqrt[3]{2} \in k$, $\zeta \notin k$	$\{1\}$	11	$\langle \tau, \iota_A\iota_B \rangle$	ac^6, ad^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$\mathbb{Z}/2\mathbb{Z}$	7	$\langle \tau, \iota_A\iota_B^3 \rangle$	ac^6, a^3d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^2$	2	$\langle \tau, \iota_A\iota_B^5 \rangle$	ac^6, a^5d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^3$	1	$\langle \tau, \iota_A^3, \iota_B^3 \rangle$	a^3c^6, b^3d^6	$a, b \notin \langle -3, k^{*2} \rangle;\ a \neq b$
	$(\mathbb{Z}/2\mathbb{Z})^4$	2	$\langle \tau, \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin \langle -3, k^{*2} \rangle$
	$\mathbb{Z}/3\mathbb{Z}$	8	$\langle \tau, \iota_A^2\iota_B^5 \rangle$	a^2c^6, a^5d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^2$	5	$\langle \tau, \iota_A^2\iota_B^2 \rangle$	a^2c^6, a^2d^6	$a \notin \langle 3, k^{*3} \rangle$
	$\mathbb{Z}/6\mathbb{Z}$	4	$\langle \tau, \iota_A \rangle$	ac^6, d^6	$a \notin \langle 3, k^{*2}, k^{*3} \rangle$
$\sqrt[3]{2} \notin k$, $\zeta \in k$	$\{1\}$	26	$\langle \sigma\iota_A^2\iota_B^2, \iota_A^3, \iota_B^3 \rangle$	$16a^3c^6, 16b^3d^6$	$a, b \notin \langle 2, k^{*2} \rangle; a \neq b$
	$(\mathbb{Z}/2\mathbb{Z})^2$	10	$\langle \sigma\iota_B^4, \iota_A^3\iota_B^3 \rangle$	$a^3c^6, 4a^3d^6$	$a \notin \langle 2, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^4$	6	$\langle \sigma, \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin \langle 2, k^{*2} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^6$	2	$\langle \sigma\iota_B^2, \iota_A^3\iota_B^3 \rangle$	$a^3c^6, 16a^3d^6$	$a \notin \langle 2, k^{*2} \rangle$
	$\mathbb{Z}/3\mathbb{Z}$	16	$\langle \sigma\iota_A^2, \iota_A^5\iota_B^2 \rangle$	$16a^5c^6, a^2d^6$	$a \notin \langle 2, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^2$	16	$\langle \sigma\iota_A\iota_B^2 \rangle$	$4a^3c^6, 16d^6$	$a \notin \langle 2, k^{*2} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^3$	4	$\langle \sigma\iota_B^2, \iota_A^2\iota_B^2 \rangle$	$a^2c^6, 16a^2d^6$	$a \notin \langle 2, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^4$	3	$\langle \sigma\iota_A^2\iota_B^2 \rangle$	$16c^6, 16d^6$	—
	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$	2	$\langle \sigma, \iota_A \rangle$	a, d^6	$a \notin \langle 2, k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/6\mathbb{Z})^2$	2	$\langle \sigma\iota_B \rangle$	$c^6, 4b^3d^6$	$b \notin \langle 2, k^{*3} \rangle$
$\sqrt[3]{2} \in k$, $\zeta \in k$	$\{1\}$	5	$\langle \iota_A\iota_B \rangle$	ac^6, ad^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^2$	5	$\langle \iota_A^3\iota_B \rangle$	a^3c^6, ad^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^4$	1	$\langle \iota_A\iota_B^5 \rangle$	ac^6, a^5d^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/2\mathbb{Z})^6$	1	$\langle \iota_A^3, \iota_B^3 \rangle$	a^3b^6, b^3d^6	$a, b \notin k^{*2};\ a \neq b$
	$(\mathbb{Z}/2\mathbb{Z})^8$	1	$\langle \iota_A^3\iota_B^3 \rangle$	a^3c^6, a^3d^6	$a \notin k^{*2}$
	$\mathbb{Z}/3\mathbb{Z}$	2	$\langle \iota_A, \iota_B^2 \rangle$	ac^6, b^2d^6	$a \notin \langle k^{*2}, k^{*3} \rangle;\ b \notin k^{*3}$
	$(\mathbb{Z}/3\mathbb{Z})^2$	3	$\langle \iota_A^5\iota_B^2 \rangle$	a^5c^6, a^2d^6	$a \notin \langle k^{*2}, k^{*3} \rangle$
	$(\mathbb{Z}/3\mathbb{Z})^4$	1	$\langle \iota_A^2\iota_B^2 \rangle$	a^2c^6, a^2d^6	$a \notin k^{*3}$
	$(\mathbb{Z}/6\mathbb{Z})^2$	2	$\langle \iota_B \rangle$	c^6, bd^6	$a \notin \langle k^{*2}, k^{*3} \rangle$

TABLE 2. Possible groups $H^1(G, \operatorname{Pic} X)$. See the proof of Theorem 1.3 for an explanation.

Proof. Let $\alpha = \beta = \sqrt[3]{4}$. By Proposition 5.1, the exceptional curves of X are defined over $K := k(\sqrt[3]{2})$, and in the notation of §6 we have $G := \mathrm{Gal}(K/k) = \langle \rho \rangle$, where $\rho = \sigma\iota_A^2\iota_B^2$. Since G is cyclic, we may apply the strategy of §3.3 by taking H to be the trivial subgroup (so $L = K$). Using the basis for $\mathrm{Pic}\, X_K \cong \mathbb{Z}^9$ of Proposition 5.2 we compute

$$\ker \overline{N}_{L/k} / \operatorname{im} \Delta \cong (\mathbb{Z}/3\mathbb{Z})^4;$$

see Table 2. The classes

$$\mathfrak{h}_1 = [(0,1,0,0,0,0,0,2,-1)], \qquad \mathfrak{h}_2 = [(0,0,0,0,1,0,0,2,-1)],$$
$$\mathfrak{h}_3 = [(0,0,0,0,0,0,1,2,-1)], \qquad \mathfrak{h}_4 = [(0,0,0,0,0,0,0,3,-1)]$$

of $\mathrm{Pic}\, X_K$ determine generators for this group.

Consider the divisor class $\mathfrak{h}_1 - \mathfrak{h}_2 = [\Gamma_2 - \Gamma_5] \in \mathrm{Pic}\, X_K$. By Theorem 3.3, this class gives a cyclic algebra $(K/k, f)$ in the image of the map $\mathrm{Br}\, X/\mathrm{Br}\, k \to \mathrm{Br}\, k(X)/\mathrm{Br}\, k$, where $f \in k(X)^*$ is any function such that $N_{K/k}(\Gamma_2 - \Gamma_5) = (f)$, that is, a function with zeroes along $\Gamma_2 + {}^{\rho}\Gamma_2 + {}^{\rho^2}\Gamma_2$ and poles along $\Gamma_5 + {}^{\rho}\Gamma_5 + {}^{\rho^2}\Gamma_5$. Using the explicit equations for Γ_2 in §5 we see that the polynomial $w + 4y^3$ vanishes along $\Gamma_2 + {}^{\rho}\Gamma_2 + {}^{\rho^2}\Gamma_2$.

Let I be the ideal of functions that vanish on $\Gamma_5, {}^{\rho}\Gamma_5$ and ${}^{\rho^2}\Gamma_5$. Explicitly,

$$I = (z - Q_5, w - C_5) \cap (z - {}^{\rho}Q_5, w - {}^{\rho}C_5) \cap (z - {}^{\rho^2}Q_5, w - {}^{\rho^2}C_5),$$

where Q_5 and C_5 are the quadratic and cubic forms, respectively, corresponding to Γ_5, and, for example, ${}^{\rho}Q_5$ is the result of applying ρ to the coefficients of Q_5. We compute a Gröbner basis for I (under the lexicographic order $w > z > y > x$) and find the polynomial $w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3$ in this basis. Hence

$$f := \frac{w + 4y^3}{w + (2\zeta + 2)zy + (-8\zeta + 4)y^3 + 12x^3}$$

has the required zeroes and poles.

Consider the following rational points of X:

$$P_1 = [1 : 0 : 0 : 4] \quad \text{and} \quad P_2 = [0 : 1 : 0 : 4].$$

Let $\mathscr{A}$ be the Azumaya algebra of X corresponding to $(K/k, f)$. Specializing the algebra $\mathscr{A}$ at P_1 we obtain the cyclic algebra $\mathscr{A}(P_1) = (K/k, 1/4)$ over k. On the other hand, specializing at P_2 we compute $\mathscr{A}(P_2) = (K/k, 1/(1-\zeta)) = (K/k, \zeta)$.

Let $\mathfrak{p}$ be the unique prime above 3 in k. To compute the invariants we observe that

$$\mathrm{inv}_{\mathfrak{p}}(\mathscr{A}(P_i)) = \frac{1}{3}[f(P_i), 2]_{\mathfrak{p}} \in \mathbb{Q}/\mathbb{Z},$$

where $[f(P_i)_{\mathfrak{p}}, 2]_{\mathfrak{p}} \in \mathbb{Z}/3\mathbb{Z}$ is the (additive) norm residue symbol. We compute $[1/4, 2]_{\mathfrak{p}} \equiv 0 \bmod 3$ (using [CTKS87, (77)]) and $[\zeta, 2]_{\mathfrak{p}} \equiv 1 \bmod 3$ (using biadditivity of the norm residue symbol and [CTKS87, (75)] with $\theta = -\zeta$, $a = 1$). Let $P \in X(\mathbb{A}_k)$ be the point that is equal to P_1 at all places except $\mathfrak{p}$, and is P_2 at $\mathfrak{p}$. Then

$$\sum_v \mathrm{inv}_v(\mathscr{A}(P_v)) = 1/3,$$

so $P \in X(\mathbb{A}_k) \setminus X(\mathbb{A}_k)^{\mathrm{Br}}$ and X is a counterexample to weak approximation.

To see that X is k-minimal, see the proof of Theorem 1.3: the surface X appears as the example in the twelfth line from the bottom of Table 2. □

Remark 7.2. The surface X of Proposition 7.1 is not birational to a conic bundle C, since the birational invariant $\operatorname{Br} X/\operatorname{Br} k$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^4$, while $\operatorname{Br} C/\operatorname{Br} k$ is always 2-torsion. In particular, the failure of weak approximation cannot be accounted for by the presence of a conic bundle structure.

7.2. **Main Theorem.** We are ready to prove our main theorem.

Proof of Theorem 1.1. Let $\alpha = \beta = \sqrt{p}$. By Proposition 5.1, the exceptional curves of X are defined over $K := \mathbb{Q}(\zeta, \sqrt[3]{2}, \sqrt{p})$, and in the notation of §6 we have $G := \operatorname{Gal}(K/\mathbb{Q}) = \langle \sigma, \tau, \iota_A^3\iota_B^3 \rangle$. One easily checks that the element $\rho := \iota_A^3\iota_B^3$ acts on exceptional curves as the Bertini involution of the surface (see §2.3).

The subgroup $H := \langle \sigma, \tau \rangle$ of G has index 2; hence it is normal and G/H is cyclic. Thus, we are in the situation described in §3.3, that is,

$$H^1(\operatorname{Gal}(L/\mathbb{Q}), \operatorname{Pic} X_L) \hookrightarrow \operatorname{Br} X/\operatorname{Br} \mathbb{Q},$$

where $L = K^H$ is $\mathbb{Q}(\sqrt{p})$ in this case. The injection is in fact an isomorphism because $H^1(H, \operatorname{Pic} X_K) = 0$, though we will not use this fact. Using the basis for $\operatorname{Pic} X_K \cong \mathbb{Z}^9$ of Proposition 5.2 we compute

$$\ker \overline{N}_{L/k}/\operatorname{im}\Delta \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The classes

$$\mathfrak{h}_1 = [(2,1,1,1,1,0,1,2,-3)] \qquad \text{and} \qquad \mathfrak{h}_2 = [(0,0,0,0,0,1,0,-1,0)] \tag{9}$$

of $\operatorname{Pic} X_L$ generate this group.

Next, we apply the procedure of the Appendix to descend the line bundle $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$ in the class of $\mathfrak{h}_2$ to a line bundle defined over $\mathbb{Q}(\sqrt{p})$. We must give isomorphisms

$$f_h : \mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8) \to \mathscr{O}_{X_K}({}^h\Gamma_6 - {}^h\Gamma_8),$$

one for each $h \in H$, satisfying the cocycle condition. In this case H is isomorphic to the symmetric group on 3 elements, with presentation

$$H = \langle \sigma, \tau \,|\, \sigma^3 = \tau^2 = 1, \sigma\tau = \tau\sigma^2 \rangle,$$

so it is enough to find isomorphisms f_σ and f_τ as above such that

$$\begin{aligned} {}^{\sigma^2}f_\sigma \circ {}^{\sigma}f_\sigma \circ f_\sigma &= id, \\ {}^{\tau}f_\tau \circ f_\tau &= id, \\ {}^{\sigma}f_\tau \circ f_\sigma &= {}^{\tau\sigma}f_\sigma \circ {}^{\tau}f_\sigma \circ f_\tau. \end{aligned}$$

For example, the map f_σ is just multiplication by a function having zeroes at Γ_6 and ${}^\sigma\Gamma_8$ and poles at Γ_8 and ${}^\sigma\Gamma_6$. We also denote this function f_σ, and find it as follows. First, take a function that vanishes on Γ_8, ${}^\sigma\Gamma_6$, and possibly some extra lines. For example, recall that

$$\Gamma_6 = V(z - Q_6(x,y), w - C_6(x,y)), \quad \text{and} \quad \Gamma_8 = V(z - Q_8(x,y), w - C_8(x,y)),$$

where Q_6 and Q_8 (resp C_6 and C_8) are the quadratic (resp. cubic) forms in x and y, corresponding to Γ_6 and Γ_8 given in §5. Let ${}^\sigma Q_6$ denote the result of applying σ to the coefficients of Q_6, and similarly for the other binary and cubic forms. The function

$$g_1 = (z - {}^\sigma Q_6(x,y))(z - Q_8(x,y))$$

vanishes on the exceptional curves[1] ${}^{\sigma}\Gamma_6$, ${}^{\sigma}\Gamma'_6$, Γ_8 and Γ'_8. Let I be the ideal of functions that vanish on Γ_6, ${}^{\sigma}\Gamma_8$, ${}^{\sigma}\Gamma'_6$ and Γ'_8. Explicitly,

$$I = (z - Q_6, w - C_6) \cap (z - {}^{\sigma}Q_8, w - {}^{\sigma}C_8) \cap (z - {}^{\sigma}Q_6, w + {}^{\sigma}C_6) \cap (z - Q_8, w + C_8).$$

We compute a Gröbner basis for I (under the lexicographic order $w > z > y > x$) and find the following degree 4 polynomial in the basis:

$$\begin{aligned} f_1 = {} & 6\sqrt{p}wy + 3\sqrt{p}(\zeta-1)(s^2+2)wx + (-2\zeta+1)sz^2 + 2p(2\zeta-1)(s^2+s+1)zy^2 \\ & + p(-\zeta-1)(3s^2+2s+2)zyx + 2p(-\zeta+2)(s^2+s+1)zx^2 + 2p^2(2\zeta-1)(s^2+1)y^4 \\ & + p^2(-\zeta-1)(3s^2+2s+2)y^3x + 2p^2(-\zeta+2)(s^2+s+1)y^2x^2 + 2p^2(2\zeta-1)(s+1)yx^3 \\ & + p^2(\zeta+1)(s^2-2)x^4. \end{aligned}$$

The function f_1/g_1 has the right zeroes and poles to be f_σ. We set

$$f_\sigma := \frac{1}{(-2\zeta+1)s} \cdot \frac{f_1}{g_1}.$$

The constant in front of f_1/g_1 is a normalization factor, making $f_\sigma([0:0:1:1]) = 1$.

Similarly, f_τ denotes a function with zeroes at Γ_6 and ${}^{\tau}\Gamma_8$ and poles at Γ_8 and ${}^{\tau}\Gamma_6$. Let

$$\begin{aligned} g_2 = {} & (z - {}^{\tau}Q_6(x,y))(z - Q_8(x,y)), \\ f_2 = {} & 6\sqrt{p}wy - \sqrt{p}wx + (-2\zeta+1)kz^2 + 2p(2\zeta-1)(k^2+k+1)zy^2 + 2p(2\zeta-1)(k+1)zyx \\ & + 2p(2\zeta-1)(k^2+k+1)zx^2 + 2p^2(2\zeta-1)(k^2+1)y^4 + 2p^2(2\zeta-1)(k+1)y^3x \\ & + 2p^2(2\zeta-1)(k^2+k+1)y^2x^2 + 2p^2(2\zeta-1)(k+1)yx^3 + 2p^2(2\zeta-1)(k^2+1)x^4. \end{aligned}$$

Then the function

$$f_\tau := \frac{1}{(-2\zeta+1)k} \cdot \frac{f_2}{g_2}$$

has zeroes at Γ_6 and ${}^{\tau}\Gamma_8$ and poles at Γ_8 and ${}^{\tau}\Gamma_6$. Because of the normalization, f_τ and f_σ satisfy the cocycle condition. Thus $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$ descends to a line bundle $\mathscr{F}$ over L, as we expected. It remains to find a divisor over L in the class of $\mathscr{F}$. To this end, we average the rational section 1 of $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$ over the group H to obtain a rational section

$$\mathfrak{s} = \sum_{h \in H} {}^{h^{-1}}(f_h) = 1 + {}^{\sigma^2}f_\sigma + {}^{\tau}f_\tau + {}^{\sigma^2}f_\sigma \cdot {}^{\sigma}f_\sigma + {}^{\sigma^2}f_\sigma \cdot {}^{\sigma}f_\sigma \cdot {}^{\sigma\tau}f_\tau + {}^{\sigma^2}f_\sigma \cdot {}^{\tau\sigma}f_\tau$$

of $\mathscr{F}$. The common denominator of $\mathfrak{s}$ is

$${}^{\sigma^2}g_1 \cdot {}^{\tau}g_2 \cdot {}^{\sigma}g_1 \cdot {}^{\sigma\tau}g_2 \cdot {}^{\tau\sigma}g_2.$$

By definition of g_1 and g_2 this denominator vanishes along the divisor

$$\begin{aligned} & 2\Gamma_6 + 2\Gamma'_6 + {}^{\sigma^2}\Gamma_8 + {}^{\sigma^2}\Gamma'_8 + {}^{\tau}\Gamma_8 + {}^{\tau}\Gamma'_8 + 2\left({}^{\sigma^2}\Gamma_6\right) + 2\left({}^{\sigma^2}\Gamma'_6\right) \\ & \quad + {}^{\sigma}\Gamma_8 + {}^{\sigma}\Gamma'_8 + {}^{\sigma\tau}\Gamma_8 + {}^{\sigma\tau}\Gamma'_8 + {}^{\sigma}\Gamma_6 + {}^{\sigma}\Gamma'_6 + {}^{\tau\sigma}\Gamma_8 + {}^{\tau\sigma}\Gamma'_8. \end{aligned}$$

Here $2\Gamma_6$ means, for example, that the denominator vanishes on this curve to order 2. The numerator of $\mathfrak{s}$ vanishes along the divisor

$$\Gamma_6 + 2\Gamma'_6 + {}^{\sigma^2}\Gamma'_8 + {}^{\tau}\Gamma'_8 + 2\left({}^{\sigma^2}\Gamma_6\right) + 2\left({}^{\sigma^2}\Gamma'_6\right) + {}^{\sigma}\Gamma'_8 + {}^{\sigma\tau}\Gamma'_8 + {}^{\sigma}\Gamma_6 + {}^{\sigma}\Gamma'_6 + {}^{\tau\sigma}\Gamma'_8 + Z,$$

[1]The notation ${}^{\sigma}\Gamma'_6$ is unambiguous (cf. Remark 4.6).

where Z is some curve on X. Thus, as a rational function, $\mathfrak{s}$ has a zero of order 1 along Z and poles of order 1 along the divisor

$$P := \Gamma_6 + {}^{\sigma^2}\Gamma_8 + {}^{\tau}\Gamma_8 + {}^{\sigma\tau}\Gamma_8 + {}^{\sigma}\Gamma_8 + {}^{\tau\sigma}\Gamma_8$$

As a *rational section* of $\mathscr{O}_{X_K}(\Gamma_6 - \Gamma_8)$, $\mathfrak{s}$ has a zero along Z and a pole along

$$P' := \Gamma_8 + {}^{\sigma^2}\Gamma_8 + {}^{\tau}\Gamma_8 + {}^{\sigma\tau}\Gamma_8 + {}^{\sigma}\Gamma_8 + {}^{\tau\sigma}\Gamma_8 = \sum_{h \in H} {}^{h}\Gamma_8.$$

The divisor $Z - P' \in \operatorname{Div} X_L$ represents the class of $\Gamma_6 - \Gamma_8$.

Let $\bar{\rho}$ be the class of ρ in $\operatorname{Gal}(L/\mathbb{Q})$. By Theorem 3.3, the class $[Z - P']$ gives a cyclic algebra $(\mathbb{Q}(\sqrt{p})/\mathbb{Q}, f)$ in $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$, where $f \in \mathbb{Q}(X)^*$ is any function such that

$$N_{\mathbb{Q}(\sqrt{p})/\mathbb{Q}}(Z - P') = Z + {}^{\bar{\rho}}Z - (P' + {}^{\bar{\rho}}P') = (f).$$

We find an explicit f. The numerator of $\mathfrak{s}$ (after cancelling out common divisors) is a polynomial of degree 12 in $\mathbb{Q}(\sqrt[3]{2}, \zeta, \sqrt{p})[x, y, z, w]$. We may express it as

$$p_1 + \sqrt[3]{2}p_2 + \sqrt[3]{4}p_3 + \zeta p_4 + \zeta\sqrt[3]{2}p_5 + \zeta\sqrt[3]{4}p_6,$$

where $p_i \in \mathbb{Q}(\sqrt{p})[x, y, z, w]$ for $i = 1, \ldots, 6$. Then $Z = V(p_1, \ldots, p_6)$. We find constants $b_i \in \mathbb{Q}(\sqrt{p})$ such that the polynomial $q = \sum_i b_i p_i$ belongs to $\mathbb{Q}[x, y, z, w]$; then the polynomial q vanishes on $Z \cup {}^{\bar{\rho}}Z$ and is a suitable numerator for f. A little linear algebra reveals that

$$\begin{aligned} q = 12z^6 &- 72pz^5y^2 - 192pz^5yx - 48pz^5x^2 + 300p^2z^4y^4 + 600p^2z^4y^3x + 576p^2z^4y^2x^2 + 408p^2z^4yx^3 \\ &+ 156p^2z^4x^4 - 288p^3z^3y^6 - 720p^3z^3y^5x - 888p^3z^3y^4x^2 - 768p^3z^3y^3x^3 - 756p^3z^3y^2x^4 - 264p^3z^3yx^5 \\ &- 204p^3z^3x^6 + 144p^4z^2y^8 + 456p^4z^2y^7x + 1032p^4z^2y^6x^2 + 1080p^4z^2y^5x^3 + 756p^4z^2y^4x^4 + 864p^4z^2y^3x^5 \\ &+ 684p^4z^2y^2x^6 + 456p^4z^2yx^7 - 48p^4z^2x^8 + 192p^5zy^{10} - 48p^5zy^9x - 720p^5zy^8x^2 - 1104p^5zy^7x^3 \\ &- 600p^5zy^6x^4 - 216p^5zy^5x^5 - 240p^5zy^4x^6 - 480p^5zy^3x^7 - 504p^5zy^2x^8 - 24p^5zyx^9 + 48p^5zx^{10} \\ &- 192p^6y^{12} - 288p^6y^{11}x + 192p^6y^{10}x^2 + 528p^6y^9x^3 + 432p^6y^8x^4 + 168p^6y^7x^5 - 192p^6y^6x^6 - 288p^6y^5x^7 \\ &+ 192p^6y^4x^8 + 312p^6y^3x^9 - 48p^6yx^{11}. \end{aligned}$$

Now we look for a polynomial r of the same degree as q vanishing on $P' + {}^{\bar{\rho}}P'$. Since ρ acts as the Bertini involution $\Gamma \mapsto \Gamma'$ on exceptional curves, we have

$$P' + {}^{\bar{\rho}}P' = \sum_{h \in H} {}^{h}(\Gamma_8 + \Gamma_8').$$

The polynomial $z - Q_8(x, y)$ vanishes on $\Gamma_8 + \Gamma_8'$. Hence we may take $r = \prod_{h \in H}(z - {}^{h}Q_8(x, y))$, and obtain

$$\begin{aligned} r = z^6 &- 6pz^5y^2 - 24pz^5yx - 6pz^5x^2 + 36p^2z^4y^4 + 78p^2z^4y^3x + 132p^2z^4y^2x^2 + 78p^2z^4yx^3 + 36p^2z^4x^4 \\ &+ 8p^3z^3y^6 - 60p^3z^3y^5x - 168p^3z^3y^4x^2 - 276p^3z^3y^3x^3 - 168p^3z^3y^2x^4 - 60p^3z^3yx^5 + 8p^3z^3x^6 - 24p^4z^2y^8 \\ &- 24p^4z^2y^7x + 156p^4z^2y^6x^2 + 396p^4z^2y^5x^3 + 540p^4z^2y^4x^4 + 396p^4z^2y^3x^5 + 156p^4z^2y^2x^6 - 24p^4z^2yx^7 \\ &- 24p^4z^2x^8 + 24p^5zy^9x + 24p^5zy^8x^2 - 120p^5zy^7x^3 - 324p^5zy^6x^4 - 432p^5zy^5x^5 - 324p^5zy^4x^6 \\ &- 120p^5zy^3x^7 + 24p^5zy^2x^8 + 24p^5zyx^9 + 16p^6y^{12} + 48p^6y^{11}x + 48p^6y^{10}x^2 + 48p^6y^9x^3 + 120p^6y^8x^4 \\ &+ 192p^6y^7x^5 + 212p^6y^6x^6 + 192p^6y^5x^7 + 120p^6y^4x^8 + 48p^6y^3x^9 + 48p^6y^2x^{10} + 48p^6yx^{11} + 16p^6x^{12}. \end{aligned}$$

Let $f = q/r$ and let $\mathscr{A}$ denote the Azumaya algebra on X corresponding to $(L/\mathbb{Q}, f)$. There are two obvious rational points on the surface X other than the anticanonical point, namely,

$$P_1 = [1 : 0 : -p : 0] \quad \text{and} \quad P_2 = [0 : 1 : -p : 0].$$

Specializing the algebra $\mathscr{A}$ at P_1 we obtain the quaternion algebra $(p, 12) \cong (p, 3)$ over $\mathbb{Q}$. The invariant of this algebra at a prime q is readily calculated using the Hilbert symbol $[\cdot\,,\cdot]_q \in \{\pm 1\}$ of the quaternion algebra (cf. [Ser73]) via the formula

$$\operatorname{inv}_q(a, b) = \frac{1 - [a, b]_q}{4} \in \mathbb{Q}/\mathbb{Z}.$$

Using the formulas for the Hilbert symbol in [Ser73], we find that

$$[p, 3]_q = \begin{cases} (-1)^{(p-1)/2} & \text{if } q = 2, \\ \left(\frac{p}{3}\right) & \text{if } q = 3, \\ \left(\frac{3}{p}\right) & \text{if } q = p, \\ 1 & \text{otherwise,} \end{cases}$$

where $\left(\frac{p}{q}\right)$ is the usual Legendre symbol. On the other hand, specializing $\mathscr{A}$ at P_2 we obtain the quaternion algebra $(p, 16) \cong (p, 1)$ over $\mathbb{Q}$. We find that $[p, 1]_q = 1$ for all primes q. Hence

$$\operatorname{inv}_3(p, 3) \neq \operatorname{inv}_3(p, 1) \text{ if } p \equiv 5 \bmod 6 \quad \text{and} \quad \operatorname{inv}_2(p, 3) \neq \operatorname{inv}_2(p, 1) \text{ if } p \equiv 3 \bmod 4. \tag{10}$$

Let $P \in X(\mathbb{A}_\mathbb{Q})$ be the point that is equal to P_1 at all places except p, and is P_2 at p. Then by (10) it follows that if $p \equiv 5 \bmod 6$ then

$$\sum_v \operatorname{inv}_v(\mathscr{A}(P_v)) = 1/2.$$

Similarly, if $P' \in X(\mathbb{A}_\mathbb{Q})$ is the point that is equal to P_1 at all places except 2, and is P_2 at 2, then by (10) we find that the sum of invariants is again $1/2$ when $p \equiv 3 \bmod 4$.

In either case, we have shown that if $p \not\equiv 1 \bmod 12$ then $X(\mathbb{A}_\mathbb{Q}) \neq X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$, and hence X does not satisfy weak approximation.

Finally, we note that $\operatorname{Pic} X = (\operatorname{Pic} X_L)^{\operatorname{Gal}(L/\mathbb{Q})} = \ker \Delta = \mathbb{Z}$, generated by the anticanonical class. In fact, $\operatorname{Pic} X_L \cong \mathbb{Z}^3$, generated by the classes (9) and the anticanonical class, and ρ acts nontrivially on the classes (9). Hence X is minimal. □

Appendix A. Galois Descent of Line Bundles

To make the isomorphism (7) explicit we need the theory of Galois descent of line bundles, which is a special case of the theory of descent of quasi-coherent sheaves over faithfully flat and quasi-compact morphisms. Good references for Galois descent are [BLR90] and [KT06]. For the general theory of descent see [Gro03].

Let K/k be a finite Galois extension of number fields. For every element $\sigma \in \operatorname{Gal}(K/k)$ let $\tilde{\sigma} : \operatorname{Spec} K \to \operatorname{Spec} K$ denote the corresponding morphism. Let X be a k-scheme, and suppose we are given a line bundle $\widetilde{\mathscr{F}}$ on the K-scheme X_K, together with a collection of isomorphisms $f_\sigma : \widetilde{\mathscr{F}} \to \tilde{\sigma}^* \widetilde{\mathscr{F}}$ such that

$$f_{\sigma\tau} = {}^\sigma f_\tau \circ f_\sigma \qquad \text{for all } \sigma, \tau \in \operatorname{Gal}(K/k), \tag{11}$$

where ${}^\sigma f_\tau := \tilde{\sigma}^* f_\tau$. Then there exists a sheaf $\mathscr{F}$ on X, and an isomorphism $\lambda \colon \mathscr{F}_K \to \widetilde{\mathscr{F}}$ such that $f_\sigma = {}^\sigma\lambda \circ \lambda^{-1}$ for all σ. Together, the equalities (11) are referred to as the cocycle condition.

If X is a geometrically integral k-scheme, then $\widetilde{\mathscr{F}} = \mathscr{O}_{X_K}(D)$ for some divisor $D \in \operatorname{Div} X_K$, and f_σ can be regarded as a function (up to multiplication by a scalar) whose associated divisor is $D - {}^\sigma D$. If $X(K) \neq \emptyset$ then one may use a point in $P \in X(K)$ to normalize the functions so that f_σ acts as the identity in the fiber of $\widetilde{\mathscr{F}}$ at P. We usually don't know if $X(K)$ is empty or not, but in the case of del Pezzo surfaces of degree 1 over k we have the anticanonical point.

To obtain a divisor for the descended line bundle, we take a rational section ξ of $\widetilde{\mathscr{F}}$ and we "average it" over the Galois group G to obtain a rational section of $\mathscr{F}$

$$\mathfrak{s} := \sum_{\sigma \in G} {}^{\sigma^{-1}}(f_\sigma(\xi)).$$

Note it may be necessary to change the choice of ξ to make $\mathfrak{s}$ nonzero. The divisor of zeroes of $\mathfrak{s}$, with respect to local trivializations for $\mathscr{F}$, gives a line bundle isomorphic to the descended line bundle. We often use the rational section $\xi = 1$, and since f_σ acts by multiplication, we obtain $\mathfrak{s} = \sum_{\sigma \in G} {}^{\sigma^{-1}}(f_\sigma)$ in this case.

References

[BSD75] B. J. Birch and H. P. F. Swinnerton-Dyer, *The Hasse problem for rational surfaces*, J. Reine Angew. Math. **274/275** (1975), 164–174. ↑1

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21, Springer, Berlin, 1990. ↑A

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. ↑1.1

[Bri02] M. Bright, *Computations on diagonal quartic surfaces*, Cambridge University, 2002. Ph. D. Thesis. ↑3.1, 3.2

[CG66] J. W. S. Cassels and M. J. T. Guy, *On the Hasse principle for cubic surfaces*, Mathematika **13** (1966), 111–120. ↑1

[CTKS87] J.-L. Colliot-Thélène, D. Kanevsky, and J.-J. Sansuc, *Arithmétique des surfaces cubiques diagonales*, Diophantine approximation and transcendence theory, 1987, pp. 1–108. ↑7.1

[CTS77] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175–229. ↑1

[CTSSD87] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer, *Intersections of two quadrics and Châtelet surfaces*, J. Reine Angew. Math. **374** (1987), 72–168. ↑2.5

[Cor05] P. Corn, *Del Pezzo surfaces and the Brauer-Manin obstruction*, University of California, Berkeley, 2005. Ph. D. Thesis. ↑3.1

[Cor07] ______, *The Brauer-Manin obstruction on del Pezzo surfaces of degree 2*, Proc. London Math. Soc. (3) **95** (2007), no. 3, 735–777. ↑3.4, 6.1

[CO99] P. Cragnolini and P. A. Oliverio, *Lines on del Pezzo surfaces with $K_S^2 = 1$ in characteristic $\neq 2$*, Comm. Algebra **27** (1999), no. 3, 1197–1206. ↑4.2, 4.5

[Dem80] M. Demazure, *Surfaces de Del Pezzo II, III, IV, V*, Séminaire sur les Singularités des Surfaces, Lecture Notes in Mathematics, vol. 777, Springer, Berlin, 1980, pp. 23–69. ↑2, 2.3, 4.5

[Ful98] W. Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete., vol. 2, Springer-Verlag, Berlin, 1998. ↑5

[Gro03] A. Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques, 3, Société Mathématique de France, Paris, 2003. ↑A

[Har04] D. Harari, *Weak approximation on algebraic varieties*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), 2004, pp. 43–60. ↑1

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 32, Springer, Berlin, 1996. ↑2, 2.2

[KT04] A. Kresch and Y. Tschinkel, *On the arithmetic of del Pezzo surfaces of degree 2*, Proc. London Math. Soc. (3) **89** (2004), no. 3, 545–569. ↑1, 1, 3.6

[KT06] ———, *Effectivity of Brauer-Manin obstructions* (December 21, 2006). Preprint math/0612665. ↑A

[KT07] ———, *Brauer-Manin obstructions to integral points* (September 7, 2007). Preprint math/0709.1154. ↑2.5

[Man74] Yu. I. Manin, *Cubic forms: algebra, geometry, arithmetic*, North-Holland Publishing Co., Amsterdam, 1974. ↑2, 2.5

[Mil80] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. ↑3

[Ser73] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer, New York, 1973. ↑7.2

[Shi90] T. Shioda, *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Paul. **39** (1990), no. 2, 211–240. ↑4.5

[Sko01] A. Skorobogatov, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001. ↑2.6

[SD62] P. Swinnerton-Dyer, *Two special cubic surfaces*, Mathematika **9** (1962), 54–56. ↑2.5

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